### Question 1

| $t$ (hours) | 0 | 1 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| $R(t)$ (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where $t$ is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where $R$ is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

(d) For $0 \le t \le 8$, is there a time $t$ when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

---

(a) $R'(2) \approx \dfrac{R(3) - R(1)}{3 - 1} = \dfrac{950 - 1190}{3 - 1} = -120$ liters/hr$^2$

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The total amount of water removed is given by $\int_0^8 R(t)\,dt$.

$$\begin{aligned}\int_0^8 R(t)\,dt &\approx 1 \cdot R(0) + 2 \cdot R(1) + 3 \cdot R(3) + 2 \cdot R(6) \\ &= 1(1340) + 2(1190) + 3(950) + 2(740) \\ &= 8050 \text{ liters}\end{aligned}$$

This is an overestimate since $R$ is a decreasing function.

$3 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{estimate} \\ 1 : \text{overestimate with reason} \end{cases}$

(c) $$\begin{aligned}\text{Total} &\approx 50000 + \int_0^8 W(t)\,dt - 8050 \\ &= 50000 + 7836.195325 - 8050 \approx 49786 \text{ liters}\end{aligned}$$

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{estimate} \end{cases}$

(d) $W(0) - R(0) > 0$, $W(8) - R(8) < 0$, and $W(t) - R(t)$ is continuous.

Therefore, the Intermediate Value Theorem guarantees at least one time $t$, $0 < t < 8$, for which $W(t) - R(t) = 0$, or $W(t) = R(t)$.

For this value of $t$, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

$2 : \begin{cases} 1 : \text{considers } W(t) - R(t) \\ 1 : \text{answer with explanation} \end{cases}$

| $t$ (hours) | 0 | 1 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| $R(t)$ (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where $t$ is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where $R$ is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

$$R'(2) \approx \frac{R(3) - R(1)}{3-1} = \frac{950-1190}{2}$$

$$R'(2) \approx -120 \ L/hr^2$$

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(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

$$\int_0^8 R(t)\,dt \approx L_4 = (1)(1340) + (2)(1190) + (3)(950) + (2)(740)$$

$$\int_0^8 R(t)\,dt \approx 8,050 \ L$$

This is an overestimate because we are taking the Left Riemann sum of a decreasing function. Therefore the left endpoints, are greater than the right endpoints in each subinterval.



Continue problem 1 on page 5.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

$t=0 \rightarrow 50{,}000$ L

Total Left $\approx 50{,}000 + \left(\int_0^8 W(t)\,dt - \int_0^8 R(t)\,dt\right)$

$\int_0^8 W(t)\,dt = 7836.19532455$

Total Left $\approx 50{,}000 + (7836.19532455 - 8050)$

$\boxed{\text{Total Left} \approx 49{,}786 \text{ L}}$

(d) For $0 \le t \le 8$, is there a time $t$ when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

$T(t) = W(t) - R(t)$

$T(0) = 2000 - 1340 = 600$

$T(8) = 81.524 - 700 = -618.47$

$\boxed{-618.47 < 0 < 600}$

When $T(t) = 0$, the rate of water being pumped in to the tank is equal to the rate of water being pumped out. Because the function $T(t)$ is continuous and differentiable, by IVT there exists some $t$ on the interval $[0, 8]$ where $T(t) = 0$.

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| $t$ (hours) | 0 | 1 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| $R(t)$ (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where $t$ is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where $R$ is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

$$R'(2) \approx \frac{950 - 1190}{3 - 1} = \boxed{-120 \text{ liters per hour}^2}$$

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

$1340 + (2)(1190) + (3)(950) + (2)(740) = 8050$

8050 liters is the total amount of water removed during the 8 hours. The left Riemann sum is an overestimate since R(t) is decreasing and the graph is below the Riemann sum.

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Continue problem 1 on page 5.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

$$\int_0^8 (W(t) - R(t))\,dt$$

$$50{,}000 + \left(\int_0^8 (W(t))\,dt - 8050\right)$$

$$50{,}000 + (7836.1953 - 8050)$$

49786.7953 liter

49,787 liters of water is in the tank.

(d) For $0 \leq t \leq 8$, is there a time $t$ when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

No there is no rate that is the same as the rate at which water is being removed because R'(t) is decreasing at a slower rate than W'(t) so the rate at which W'(t) is decreasing is much greater than R'(t).

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$\frac{y_2 - y_1}{x_2 - x_1}$

| x $t$ (hours) | 0 | 1 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| y $R(t)$ (liters / hour) | 1340 | 1190 | 950 | 740 | 700 |

1. Water is pumped into a tank at a rate modeled by $W(t) = 2000e^{-t^2/20}$ liters per hour for $0 \le t \le 8$, where $t$ is measured in hours. Water is removed from the tank at a rate modeled by $R(t)$ liters per hour, where $R$ is differentiable and decreasing on $0 \le t \le 8$. Selected values of $R(t)$ are shown in the table above. At time $t = 0$, there are 50,000 liters of water in the tank.

(a) Estimate $R'(2)$. Show the work that leads to your answer. Indicate units of measure.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1190 - 1340}{1 - 0} = -150 \ \frac{L^2}{hr}$$

(b) Use a left Riemann sum with the four subintervals indicated by the table to estimate the total amount of water removed from the tank during the 8 hours. Is this an overestimate or an underestimate of the total amount of water removed? Give a reason for your answer.

$$1(1340) + 2(1190) + 3(950) + 2(740) = 21370 \ \frac{\text{liters}}{\text{hour}}$$

over estimate b/c left Riemann sum.



Continue problem 1 on page 5.

(c) Use your answer from part (b) to find an estimate of the total amount of water in the tank, to the nearest liter, at the end of 8 hours.

$50,000 + \int_0^8 w(t)\,dt - 21370 = 36466$ Liters in the tank after 8 hours

(d) For $0 \le t \le 8$, is there a time $t$ when the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank? Explain why or why not.

yes b/c the ammount of water pumped out is increasing.

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## Question 1

### Overview

In this problem students were given a function $W$ that models the rate, in liters per hour, at which water is pumped into a tank at time $t$ hours. They were also given a function $R$ that models the rate, in liters per hour, at which water is removed from the tank. $W$ is defined as an exponential function on the interval $0 \le t \le 8,$ and $R$ is differentiable and decreasing on $0 \le t \le 8.$ Selected values of $R(t)$ are given in a table. The amount of water in the tank, in liters, at time $t = 0$ is given. In part (a) students needed to estimate $R'(2)$ by calculating the value of an appropriate difference quotient based on the values in the table. Units of $\text{liters/hr}^2$ are required. In part (b) students needed to use a left Riemann sum approximation for $\int_0^8 R(t)\,dt$ to estimate the total amount of water removed from the tank during the interval $0 \le t \le 8.$ Students needed to use the appropriate function values from the table with the four subintervals $[0, 1],$ $[1, 3],$ $[3, 6],$ and $[6, 8].$ By applying the given information that $R$ is decreasing, students needed to conclude that the left Riemann sum approximation is an overestimate. In part (c) students needed to estimate the total amount of water in the tank at time $t = 8.$ This required adding the amount of water in the tank at time $t = 0$ to the amount of water pumped into the tank during the interval $0 \le t \le 8,$ and then subtracting the overestimate found in part (b). The definite integral $\int_0^8 W(t)\,dt$ gives the amount of water pumped into the tank during the interval $0 \le t \le 8$ and is evaluated using the calculator. In part (d) students needed to apply the Intermediate Value Theorem to $W(t) - R(t).$ This theorem guarantees at least one time $t$ on the interval $0 < t < 8$ for which $W(t) - R(t) = 0$ or $W(t) = R(t).$ For this value of $t$, the rate at which water is pumped into the tank is the same as the rate at which water is removed from the tank.

**Sample: 1A**
**Score: 9**

The response earned all 9 points.

**Sample: 1B**
**Score: 6**

The response earned 6 points: 2 points in part (a), 3 points in part (b), 1 point in part (c), and no points in part (d). In parts (a) and (b) the student's work is correct. In part (c) the student earned the first point for the correct definite integral. The student did not earn the second point for the estimate because of an arithmetic error. In part (d) the student earned no points.

**Sample: 1C**
**Score: 3**

The response earned 3 points: no points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the student calculates an estimate using an incorrect interval, and the units are incorrect. In part (b) the student has a correct left Riemann sum and earned the first point. The student has an incorrect estimate and does not support the answer of an overestimate with a valid reason. In part (c) the student earned the first point for the definite integral. The estimate is consistent with the student's estimate from part (b), so the second point was earned. In part (d) the student earned no points.

## Question 2

For $t \ge 0,$ a particle moves along the $x$-axis. The velocity of the particle at time $t$ is given by $v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right).$ The particle is at position $x = 2$ at time $t = 4.$

(a) At time $t = 4,$ is the particle speeding up or slowing down?

(b) Find all times $t$ in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.

(c) Find the position of the particle at time $t = 0.$

(d) Find the total distance the particle travels from time $t = 0$ to time $t = 3.$

---

| | |
|---|---|
| (a) $v(4) = 2.978716 > 0$<br>$v'(4) = -1.164000 < 0$<br><br>The particle is slowing down since the velocity and acceleration have different signs. | 2 : conclusion with reason |
| (b) $v(t) = 0 \Rightarrow t = 2.707468$<br><br>$v(t)$ changes from positive to negative at $t = 2.707.$ Therefore, the particle changes direction at this time. | $2 : \begin{cases} 1 : t = 2.707 \\ 1 : \text{justification} \end{cases}$ |
| (c) $x(0) = x(4) + \int_4^0 v(t)\,dt$<br>$= 2 + (-5.815027) = -3.815$ | $3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$ |
| (d) Distance $= \int_0^3 \lvert v(t) \rvert\,dt = 5.301$ | $2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$ |

2. For $t \geq 0$, a particle moves along the $x$-axis. The velocity of the particle at time $t$ is given by $v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right)$. The particle is at position $x = 2$ at time $t = 4$.

(a) At time $t = 4$, is the particle speeding up or slowing down?

$v(4) = 1 + 2\sin\left(\frac{4^2}{2}\right)$

$v(4) = 2.979$

$v'(t) = 2\cos\left(\frac{t^2}{2}\right)(t)$

$v'(4) = 2\cos\left(\frac{4^2}{2}\right)(4)$

$v'(4) = -1.164$

Slowing down because $v(4)$ is positive and $v'(4)$ is negative.

---

(b) Find all times $t$ in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.

$1 + 2\sin\left(\frac{t^2}{2}\right) = 0$

$t = 2.707$

The particle changes direction one time at $t = 2.707$ because $v(t) = 0$ and $v(t)$ changes from positive to negative.



Continue problem 2 on page 7.





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(c) Find the position of the particle at time $t = 0$.

$2 + \int_4^0 v(t)\,dt = \boxed{-3.815}$

(d) Find the total distance the particle travels from time $t = 0$ to time $t = 3$.

$\int_0^3 |v(t)|\,dt = \boxed{5.301}$

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2. For $t \geq 0$, a particle moves along the $x$-axis. The velocity of the particle at time $t$ is given by $v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right)$. The particle is at position $x = 2$ at time $t = 4$.

(a) At time $t = 4$, is the particle speeding up or slowing down?

a(t) & v(t) same or diff sign?

$v(4) = 1 + 2\sin\left(\frac{4^2}{2}\right) = 2.979$ $+$

$a(t) = 2\cos\left(\frac{t^2}{2}\right) \cdot t$

$a(4) = -1.164$ $-$

particle is slowing down at $t=4$ b/c a(t) & v(t) have different signs

(b) Find all times $t$ in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.

$v(t) = 0$

$1 + 2\sin\left(\frac{t^2}{2}\right) = 0$

$t = 2.707468$

particle changes direction at $t = 2.707$ because the velocity changes sign at that time

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Continue problem 2 on page 7.

(c) Find the position of the particle at time $t = 0$.

position: $\int v(t)\,dt$

$p = \int 1 + 2\sin\left(\frac{t^2}{2}\right) dt \quad u = \frac{1}{2}t^2 \quad du = t\,dt$

$\frac{1}{t}du = dt$

$pos = \frac{1}{t}\int 1 + 2\sin(u)\,du$

$\frac{1}{t}\left(t - 2\cos u\right) + c$

$\frac{1}{t}\left(t - 2\cos\left(\frac{t^2}{2}\right)\right)$

(d) Find the total distance the particle travels from time $t = 0$ to time $t = 3$.

total distance $= \int_0^3 |v(t)|\,dt$

$\int_0^3 \left|1 + 2\sin\left(\frac{t^2}{2}\right)\right| dt = \boxed{5.301}$

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2. For $t \geq 0$, a particle moves along the $x$-axis. The velocity of the particle at time $t$ is given by $v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right)$. The particle is at position $x = 2$ at time $t = 4$.

(a) At time $t = 4$, is the particle speeding up or slowing down?

At time t=4 the particle is slowing down

(b) Find all times $t$ in the interval $0 < t < 3$ when the particle changes direction. Justify your answer.

$v(t) = 1 + 2\sin\left(\frac{t^2}{2}\right) = 0$

$v'(t) = t + 2t^2\cos\left(\frac{t^2}{2}\right) = 0$

particle changes direction at t = 2.607 and t = 1.375

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Continue problem 2 on page 7.

(c) Find the position of the particle at time $t = 0$.

$$\int_4^0 \left(1+2\sin\left(\frac{t^2}{2}\right)\right) dt$$

$$= -\int_0^4 \left(1+2\sin\left(\frac{t^2}{2}\right)\right) dt$$

$$= -5.815$$

(d) Find the total distance the particle travels from time $t = 0$ to time $t = 3$.

$$\int_0^3 \left|\left(1+2\sin\left(\frac{t^2}{2}\right)\right)\right| dt$$

$$= 5.301$$

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## Question 2

### Overview

In this problem students were given information about a particle moving along the $x$-axis for time $t \geq 0$. The velocity of the particle is given as a trigonometric function, and the particle is at position $x = 2$ at time $t = 4$. In part (a) students needed to conclude that the particle is slowing down at $t = 4$ because $v(4)$ and $v'(4)$ have different signs. In part (b) students needed to determine when the particle changes direction in the interval $0 < t < 3$, and justify their answer. This required use of the calculator to solve $v(t) = 0$ on $0 < t < 3$. In part (c) students needed to apply the Fundamental Theorem of Calculus to find the position of the particle at time $t = 0$; i.e., $x(0) = x(4) - \int_0^4 v(t)\,dt$. The expression is evaluated using the calculator. In part (d) students needed to find the total distance the particle travels from $t = 0$ to $t = 3$. Students were expected to set up and evaluate $\int_0^3 |v(t)|\,dt$ (or an appropriate sum of definite integrals) using the calculator.

**Sample: 2A**
**Score: 9**

The response earned all 9 points.

**Sample: 2B**
**Score: 6**

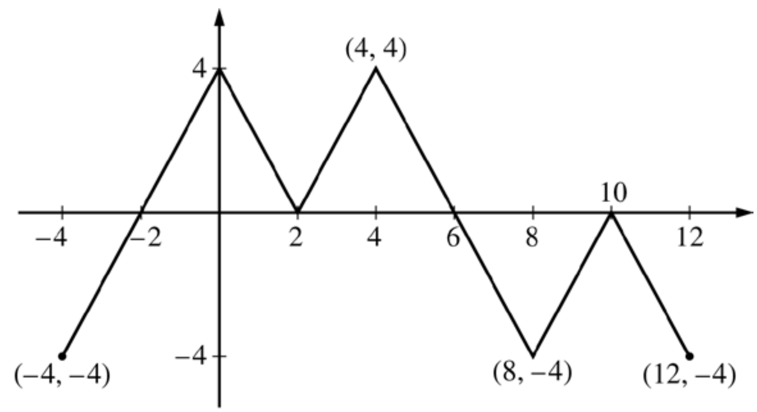
The response earned 6 points: 2 points in part (a), 2 points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student's work is correct. The student is not required to explicitly state that $a(4) = v'(4)$. In part (b) the student's work is correct. In part (c) the student is not working with a definite integral and did not earn the first point. The student was not eligible to earn the other 2 points. In part (d) the student's work is correct.

**Sample: 2C**
**Score: 3**

The response earned 3 points: no points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student has a conclusion without a reason, so no points were earned. In part (b) the student reports two incorrect values of $t$. The student did not earn the first point and was not eligible for the second point. In part (c) the student earned the first point for a correct definite integral. The student does not use the initial condition and was not eligible to earn the other 2 points. In part (d) the student's work is correct.

## Question 3

The figure above shows the graph of the piecewise-linear function $f$. For $-4 \le x \le 12$, the function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$.



Graph of $f$

(a) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of $g$ have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of $g$ on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

---

1 : $g'(x) = f(x)$ in (a), (b), (c), or (d)

(a) The function $g$ has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

1 : answer with justification

(b) The graph of $g$ has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

1 : answer with justification

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

| $x$ | $g(x)$ |
|---|---|
| −4 | −4 |
| −2 | −8 |
| 6 | 8 |
| 12 | −4 |

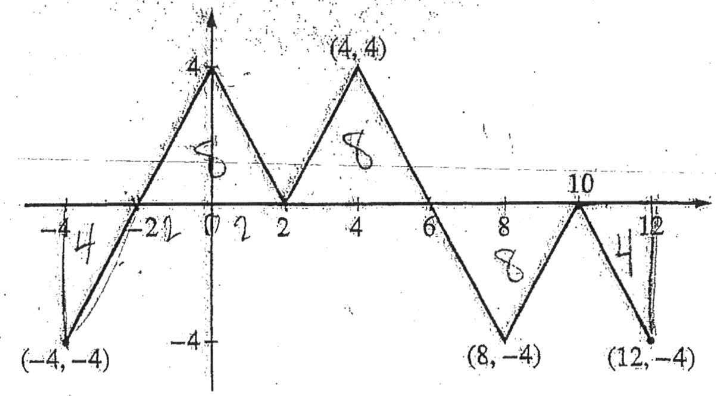
On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

$$4 : \begin{cases} 1 : \text{considers } x = -2 \text{ and } x = 6 \text{ as candidates} \\ 1 : \text{considers } x = -4 \text{ and } x = 12 \\ 2 : \text{answers with justification} \end{cases}$$

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

2 : intervals

**NO CALCULATOR ALLOWED**




Graph of $f$

3. The figure above shows the graph of the piecewise-linear function $f$. For $-4 \le x \le 12$, the function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 10$ ? Justify your answer.

$g'(x) = f(x)$

g does not have a relative minimum or maximum at $x=10$ because $g'(x) = f(x)$ does not change sign at this point

(b) Does the graph of $g$ have a point of inflection at $x = 4$ ? Justify your answer.

$g''(x) = f'(x)$

$f'(x) = g''(x)$ does change sign at $x=4$ so g does have a point of inflection at this point



Continue problem 3 on page 15.

(c) Find the absolute minimum value and the absolute maximum value of $g$ on the interval $-4 \le x \le 12$. Justify your answers.

$g'(x) = f(x) = 0 \quad x = -2 \quad x = 2$
$x = 6 \quad x = 10$

does not change sign at $x = 2$ and $x = 10$

| $x$ | $g(x)$ |
|---|---|
| $-4$ | $\int_2^{-4} f(t)\,dt = -8 + 4 = -4$ |
| $-2$ | $\int_2^{-2} f(t)\,dt = -8$ |
| $6$ | $\int_2^{6} f(t)\,dt = 8$ |
| $12$ | $\int_2^{12} f(t)\,dt = 8 - 8 - 4 = -4$ |

The absolute minimum value of $g$ on the interval $-4 \le x \le 12$ is $-8$ and the absolute maximum value of $g$ is $8$

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$g(x) = \int_2^x f(t)\,dt \le 0$

$g(x) = 0$ at $x = 2$ and $x = 10$

$g(x) \le 0$ in the intervals $-4 \le x \le 2$ and $10 \le x \le 12$







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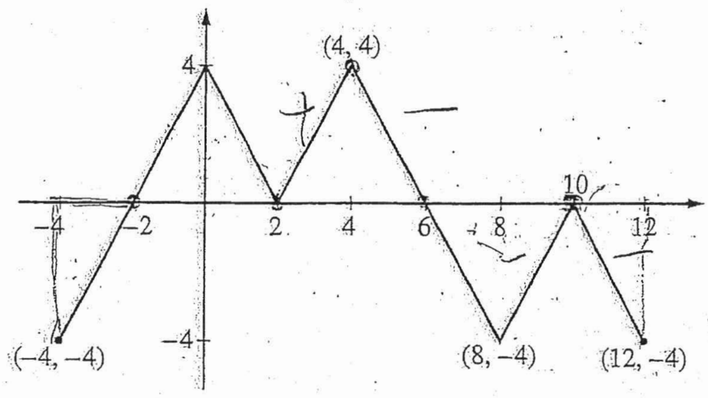
**NO CALCULATOR ALLOWED**




Graph of $f$

3. The figure above shows the graph of the piecewise-linear function $f$. For $-4 \le x \le 12$, the function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 10$ ? Justify your answer.

$g'(x) = f(x)$

$g'(10) = f(10) = 0$

g is neither at x=10 bc g'(x) does not change from pos to neg or neg to pos at x = 10.

(b) Does the graph of $g$ have a point of inflection at $x = 4$ ? Justify your answer.

$g''(x) = f'(x)$

$f'(4) = 0$

g has a poi at x=4 bc $g''(4) = 0$ and $g''(x) > 0$ when $2 \le x < 4$ and $g''(x) < 0$ when $4 < x < 8$.



Continue problem 3 on page 15.

**NO CALCULATOR ALLOWED**

(c) Find the absolute minimum value and the absolute maximum value of $g$ on the interval $-4 \le x \le 12$. Justify your answers.

$g'(x) = f(x) = 0$

$x = -2, 2, 6, 10$

| $x$ | $g(x)$ |
|---|---|
| $-4$ | $-\int_{-4}^{2} f(t)\,dt = -\left[(\tfrac{1}{2})(-4)(2) + (\tfrac{1}{2})(4)(4)\right] = -(-4+8) = -4$ |
| $-2$ | $-\int_{-2}^{2} f(t)\,dt = -\left[(\tfrac{1}{2})(4)(4)\right] = -8$ |
| $2$ | $\int_{2}^{2} f(t)\,dt = 0$ |
| $6$ | $\int_{2}^{6} f(t)\,dt = (\tfrac{1}{2})(4)(4) = 8$ |
| $10$ | $\int_{2}^{10} f(t)\,dt = 0$ |
| $12$ | $\int_{2}^{12} f(t)\,dt = (\tfrac{1}{2})(-4)(2) = -4$ |

abs max $\rightarrow x = 6$ abs min $\Rightarrow x = -2$

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$\int_{2}^{x} f(t)\,dt \le 0$

$\boxed{(10, 12) \quad \cancel{(-4, 2)}}$







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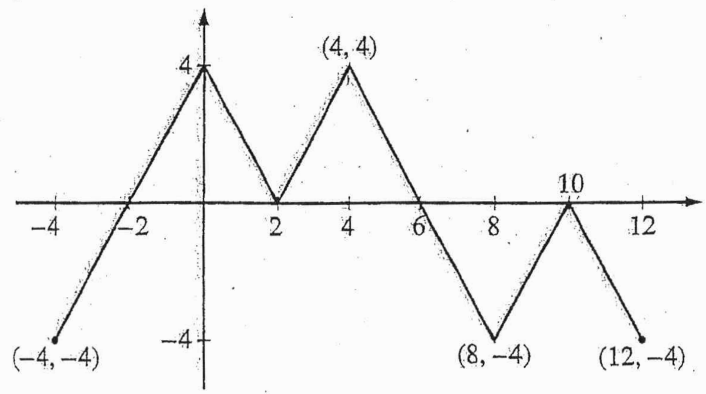
**NO CALCULATOR ALLOWED**




Graph of $f$

3. The figure above shows the graph of the piecewise-linear function $f$. For $-4 \le x \le 12$, the function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does $g$ have a relative minimum, a relative maximum, or neither at $x = 10$ ? Justify your answer.

$g'(x) = f(x)$

Since $g'(x) = f(x)$, the graph of $g$ has a relative maximum at $x=10$ because the graph of $f$ increaces before $x = 10$ and decreaces after $x=10$ and $x = 10$ is a critical point.

(b) Does the graph of $g$ have a point of inflection at $x = 4$ ? Justify your answer.

$g''(x) = f'(x)$

Since $g''(x) = f'(x)$, the graph of $g$ has an inflection point at $x=4$ because the graph of $f$ increaces before $x = 4$ and decreaces after $x = 4$.

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Continue problem 3 on page 15.

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**NO CALCULATOR ALLOWED**

(c) Find the absolute minimum value and the absolute maximum value of $g$ on the interval $-4 \le x \le 12$. Justify your answers.

$g'(x) = f(x) = 0$

$x = -2, 2, 6, 10$

absolute maximum at $x = 10$ and absolute minimum at $x = 2$

. The absolute value for both extremas are 0 since

It is found by $g(x) = f(x) = 0$.

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(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.

$g(x)$ is decreasing when $g'(x)$ is $\le 0$ and $g''(x)$ is $\le 0$

since $g'(x) = f(x)$ and $g''(x) = f'(x)$, we know that

$6 < x < 10$ and $10 < x < 12$ are the only intervals

where both $g(x)$ and $g''(x)$, which is $f(x)$ and $f'(x)$,

are decreasing (having the same sign).



GO ON TO THE NEXT PAGE.

## Question 3

### Overview

In this problem students were given the graph of $f$, a piecewise-linear function defined on the interval $[-4, 12]$. A second function $g$ is defined by $g(x) = \int_2^x f(t)\,dt$. In part (a) students needed to determine whether $g$ has a relative minimum, a relative maximum, or neither at $x = 10$, and justify their answer. Using the Fundamental Theorem of Calculus, students needed to recognize that $g'(x) = f(x)$ for all $x$ in the interval $[-4, 12]$. Since $g'(10) = f(10) = 0$ and $f(x) \le 0$ for $[8, 12]$, the First Derivative Test may be applied to conclude that there is no relative extremum at $x = 10$. In part (b) students needed to determine whether the graph of $g$ has a point of inflection at $x = 4$, and justify their answer. Since $g'(x) = f(x)$, the graph of $g$ has a point of inflection at $x = 4$ because $f$ changes from increasing to decreasing at $x = 4$. In part (c) students needed to find the absolute minimum value and the absolute maximum value of $g$ on $[-4, 12]$. Since $g'(x) = f(x)$, students were expected to find relative extrema of $g$ by identifying $x$-values where $f$ changes sign. The absolute extrema occur either at the endpoints of the interval or at the relative extrema. By comparing the values of $g$ at the four candidate $x$-values, students choose and justify the absolute extrema. Properties of the definite integral and the relation of the definite integral to accumulated area must be used to find the values of $g$. In part (d) students needed to find all intervals in $[-4, 12]$ for which $g(x) \le 0$. This part also required properties of the definite integral and the relation of the definite integral to accumulated area.

**Sample: 3A**
**Score: 9**

The response earned all 9 points. The student earned the $g'(x) = f(x)$ point in part (a). In part (a) the student earned the point with justification "$g'(x) = f(x)$ does not change sign at this point." In part (b) the student earned the point with justification "$f'(x) = g''(x)$ does change sign at $x = 4$." In part (c) the student identifies the absolute minimum and absolute maximum values with a candidates test that uses the necessary critical points. In part (d) the student gives the two correct closed intervals.

**Sample: 3B**
**Score: 6**

The response earned 6 points: 1 point for $g'(x) = f(x)$, 1 point in part (a), no points in part (b), 3 points in part (c), and 1 point in part (d). The student earned the $g'(x) = f(x)$ point in part (a). In part (a) the student earned the point with justification "$g'(x)$ does not change from pos to neg or neg to pos at $x = 10$." In part (b) the student gives the correct answer but includes an incorrect statement that $g''(4) = 0$. In part (c) the student earned the first 2 points. The student does not identify the absolute minimum as $-8$ or the absolute maximum as 8. The student earned 1 of the 2 answers with justification points. In part (d) the student does not include the endpoints of the intervals, so 1 point was earned.

**Sample: 3C**
**Score: 3**

The response earned 3 points: 1 point for $g'(x) = f(x)$, no points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). The student earned the $g'(x) = f(x)$ point in part (a). In part (a) the student has

## Question 3 (continued)

an incorrect answer. In part (b) the student's work is correct. In part (c) the student earned the first point by identifying $x = -2$ and $x = 6$ in the second line. The student earned no other points. In part (d) the student has an incorrect interval $(6, 10)$ that has no values where $g(x) \leq 0$.

## Question 4

Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

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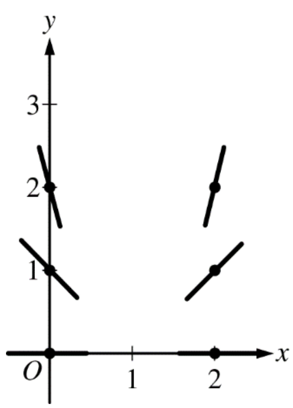
(a)




$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) $\left.\frac{dy}{dx}\right|_{(x,\,y)=(2,\,3)} = \frac{3^2}{2-1} = 9$

An equation for the tangent line is $y = 9(x-2) + 3$.

$f(2.1) \approx 9(2.1-2) + 3 = 3.9$

$2 : \begin{cases} 1 : \text{tangent line equation} \\ 1 : \text{approximation} \end{cases}$

(c) $\frac{1}{y^2}\,dy = \frac{1}{x-1}\,dx$

$\int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx$

$-\frac{1}{y} = \ln|x-1| + C$

$-\frac{1}{3} = \ln|2-1| + C \Rightarrow C = -\frac{1}{3}$

$-\frac{1}{y} = \ln|x-1| - \frac{1}{3}$

$y = \frac{1}{\frac{1}{3} - \ln(x-1)}$

Note: This solution is valid for $1 < x < 1 + e^{1/3}$.

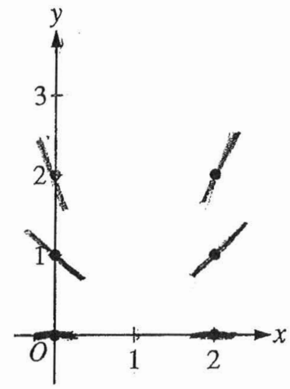
$5 : \begin{cases} 1 : \text{separation of variables} \\ 2 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad\ \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/5$ [1-2-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.



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(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}\Bigg|_{(x,y)=(2,3)} = \frac{9}{2-1} = \frac{9}{1} = 9$$

$$y - 3 = 9(x-2)$$

$$y = 9x - 18 + 3 = 9x - 15$$

$$y = 9(2.1) - 15 = 18.9 - 15 = 3.9$$



Continue problem 4 on page 17.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$\int \frac{dy}{y^2} = \int \frac{dx}{x-1}$$

$$\frac{y^{-1}}{-1} = \ln|x-1| + C$$

$$\frac{-1}{y} = \ln|x-1| + C$$

$$\frac{-1}{3} = \ln|2-1| + C$$

$$\frac{-1}{3} = \overset{0}{\cancel{\ln 1}} + C$$

$$C = \frac{-1}{3}$$

$$\frac{-1}{y} = \ln(x-1) + \left(-\frac{1}{3}\right)$$

$$-1 = \left(\ln(x-1) - \frac{1}{3}\right) y$$

$$y = \frac{-1}{\ln(x-1) - \frac{1}{3}}$$

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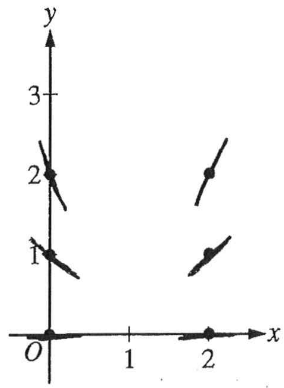
4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.




$\frac{0^2}{0-1} = 0$

$\frac{1}{0-1} = -1$

$\frac{4}{0-1} = -4$

$\frac{1}{2-1} = 1$ $\frac{0}{2-1} = 0$

$\frac{4}{2-1} = 4$

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

$\frac{dy}{dx} = \frac{y^2}{x-1}$

$y - 3 = 9(x-2)$

$\frac{dy}{dx} = \frac{9}{2-1} = 9$

$L(a) = f(a) + f'(a)(x-a)$

$3 + 9(2.1-2)$

$3 + 9(.1)$

$f(2.1) \approx 3.9$



Continue problem 4 on page 17.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$$\frac{dy}{dx} = \frac{y^2}{x-1}$$

$$dy = \frac{y^2 dx}{x-1}$$

$$(x-1)^{-1} dy = y^2 dx$$

$$\int y^{-2} dy = \int (x-1) dx$$

$$-y^{-1} = \frac{1}{2}x^2 - x + C$$

$$-\frac{1}{3} = \frac{1}{2}(2)^2 - 2 + C$$

$$-\frac{1}{3} = 2 - 2 + C$$

$$C = -\frac{1}{3}$$

$$-\frac{1}{y} = \frac{1}{2}x^2 - x - \frac{1}{3}$$

$$-1 = \left(\frac{1}{2}x^2 - x - \frac{1}{3}\right)y$$

$$y = \frac{-1}{\frac{1}{2}x^2 - x - \frac{1}{3}}$$

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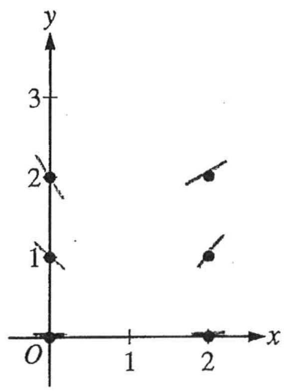
4. Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

$\frac{0^2}{0-1} = \frac{0}{-1} = 0$

$\frac{1}{-1} = -1$

$\frac{4}{-1} = -4$




$\frac{1}{2-1} = 1$

$\frac{4}{2-1} = \frac{1}{2}$

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

$x = 2 \quad y = 3$

$\frac{y^2}{x-1} \rightarrow \frac{3^2}{2-1} = \frac{9}{1} = 9 = m$

$y - y_1 = m(x - x_1)$

$y - 3 = 9(x - 2)$

$y = 9x - 18 + 3$

$\boxed{y = 9x - 15}$

$f(2.1) \approx 9(2.1) - 15$

$= 3.9$

$$\begin{array}{r} 2.1 \\ \times \; 9 \\ \hline 18.9 \\ -15.0 \\ \hline 3.9 \end{array}$$

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Continue problem 4 on page 17.

**NO CALCULATOR ALLOWED**



(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

$x = 2 \quad y = 3$

$$\frac{dy}{dx} = \frac{y^2}{x-1} \qquad (x-1)(dy) = (y^2)(dx)$$

$$\frac{(x-1)}{dx} = \frac{y^2}{dy}$$

$$\int x-1 \cdot \frac{1}{dx} = \int y^2 \cdot \frac{1}{dy}$$

$$\frac{1}{2}x^2 - x = \frac{1}{3}y^3 + c$$

$$c: \quad \frac{1}{2}(2)^2 - 2 = \frac{1}{3}(3)^3 + c$$

$$0 = 9 + c \qquad c = -9$$

$$\frac{1}{2}x^2 - x = \frac{1}{3}y^3 - 9$$

$$\frac{1}{2}x^2 - x + 9 = \frac{1}{3}y^3$$

$$3\left(\frac{1}{2}x^2 - x + 9\right) = y^3$$

$$\frac{3}{2}x^2 - 3x + 27 = y^3$$

$$\sqrt[3]{3/2\,x^2 - 3x + 27} = y$$

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## Question 4

**Overview**

In this problem students were presented with a first-order separable differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$. In part (a) students needed to sketch a slope field at six points in the *xy*-plane provided: $(0, 0)$, $(0, 1)$, $(0, 2)$, $(2, 0)$, $(2, 1)$, and $(2, 2)$. In part (b) students were given that $y = f(x)$ is the particular solution to the differential equation with the initial condition $f(2) = 3$. Students needed to write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$, where the slope is computed using the given $\frac{dy}{dx}$. The value of $f(2.1)$ is approximated using the tangent line. In part (c) students were expected to use separation of variables to find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 3$.

**Sample: 4A**
**Score: 9**

The response earned all 9 points.

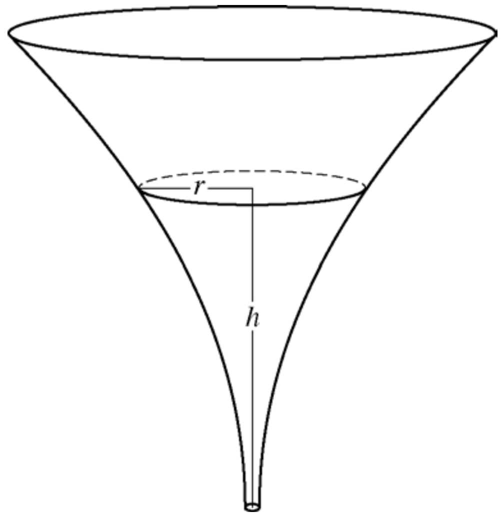
**Sample: 4B**
**Score: 6**

The response earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In parts (a) and (b) the student's work is correct. In part (c) the student incorrectly separates the differential equation. The student was eligible for and earned 1 of the 2 antiderivatives points for the correct antidifferentiation of $y^{-2}\,dy$. This side of the equation is consistent with a correct separation of variables. Because the student earned at least 1 of the first 3 points, the student was eligible for and earned the fourth point. The student was not eligible for the last point.

**Sample: 4C**
**Score: 3**

The response earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student earned the first point. The student has an inconsistent slope at $(2, 2)$, so the second point was not earned. In part (b) the student's work is correct. In part (c) the student does not have a correct approach for separation of variables. The student was not eligible for any points.

## Question 5




The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height $h$, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of $r$ and $h$ are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

---

(a) Average radius $= \frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right) dh = \frac{1}{200}\left[3h + \frac{h^3}{3}\right]_0^{10}$

$$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - 0\right) = \frac{109}{60} \text{ in}$$

$3 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi\int_0^{10} \left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2 dh = \frac{\pi}{400}\int_0^{10} \left(9 + 6h^2 + h^4\right) dh$

$$= \frac{\pi}{400}\left[9h + 2h^3 + \frac{h^5}{5}\right]_0^{10}$$

$$= \frac{\pi}{400}\left(\left(90 + 2000 + \frac{100000}{5}\right) - 0\right) = \frac{2209\pi}{40} \text{ in}^3$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$

$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$

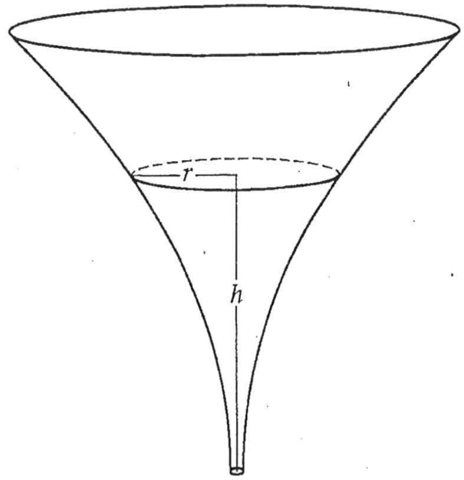
$\frac{dh}{dt} = -\frac{1}{5} \cdot \frac{10}{3} = -\frac{2}{3}$ in/sec

$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

**NO CALCULATOR ALLOWED**

1 of 2

5A




5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height $h$, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of $r$ and $h$ are inches.

(a) Find the average value of the radius of the funnel.

$$r_{avg} = \frac{1}{200}\int_0^{10} (3+h^2)\,dh$$

$$= \frac{1}{200}\left[\left(3h+\frac{h^3}{3}\right)\Big|_0^{10}\right]$$

$$= \frac{1}{200}\left(30+\frac{1000}{3}\right)$$

$$= \frac{109}{60} \text{ inches.}$$

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Continue problem 5 on page 19.

**NO CALCULATOR ALLOWED**

2 of 2

5A

(b) Find the volume of the funnel.

$$V = \frac{\pi}{400}\int_0^{10} (3+h^2)^2\, dh$$

$$= \frac{\pi}{400}\int_0^{10} (9 + 6h^2 + h^4)\, dh$$

$$= \frac{\pi}{400}\left(9h + 2h^3 + \frac{h^5}{5}\right)\Big|_0^{10}\Big]$$

$$= \frac{\pi}{400}(90 + 2000 + 20000)$$

$$= \frac{2209\pi}{40} \text{ in}^3$$

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$r = \frac{1}{20}(3 + h^2)$$

$$\frac{dr}{dt} = \left(\frac{h}{10}\right)\frac{dh}{dt}$$

$$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{2}{3} \text{ inch per second}$$

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 1 of 2

**5B**

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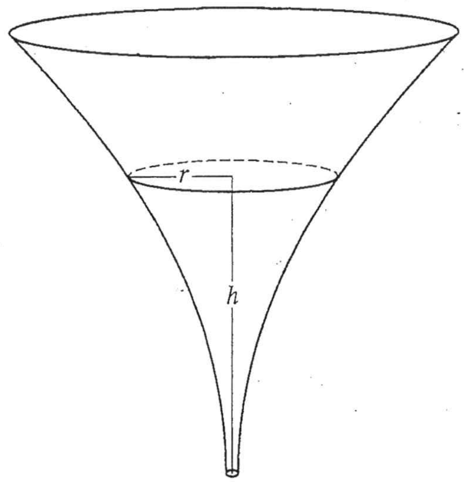


5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height $h$, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of $r$ and $h$ are inches.

(a) Find the average value of the radius of the funnel.

$$\frac{1}{10}\int_0^{10} \frac{1}{20}\left(3+h^2\right)dh = \frac{1}{200}\int_0^{10}\left(3+h^2\right)dh$$

$$= \frac{1}{200}\left(3h + \frac{1}{3}h^3\right) + C\Big]_0^{10}$$

$$= \frac{1}{200}\left(\left(30 + \frac{1000}{3}\right) - (0)\right) \text{ inches}$$





Continue problem 5 on page 19.

**NO CALCULATOR ALLOWED**

5B

(b) Find the volume of the funnel.

$$\pi \int_0^{10} \frac{1}{20}(3+h^2)\,dh = V$$

$$\frac{\pi}{20}\left(3h + \frac{1}{3}h^3\right) + C \Big]_0^{10} = \frac{\pi}{20}\left(30 + \frac{1000}{3}\right) \text{ cubic inches}$$

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$\frac{dr}{dt} = \frac{1}{20}(2h)\left(\frac{dh}{dt}\right)$$

$$-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt} \qquad \frac{dh}{dt} = -\frac{10}{5(3)} \text{ inches per second}$$

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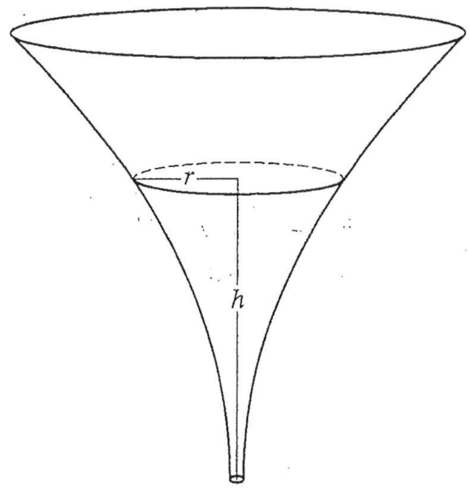


5. The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height $h$, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of $r$ and $h$ are inches.

(a) Find the average value of the radius of the funnel.

$r = \frac{1}{20}(3+h^2)$. where $0 \le h \le 10$.

$$\bar{r} = \frac{r(10) - r(0)}{10-0} = \frac{\frac{103}{20} - \frac{3}{20}}{10} = \frac{1}{2} \text{ (inches)}.$$

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Continue problem 5 on page 19.

2 of 2

5C

NO CALCULATOR ALLOWED

(b) Find the volume of the funnel.

$$V = \int_0^{10} \pi r^2 \, dh = \pi \int_0^{10} \left(\frac{3+h^2}{20}\right)^2 dh$$

$$= \pi \frac{1}{6h}\left(\frac{3+h^2}{20}\right)^3 \Big]_0^{10}$$

$$= \pi \cdot \frac{1}{60}\left(\frac{103}{20}\right)^2 - 0$$

$$= \frac{\pi}{60} \cdot \frac{103^2}{400}$$

$$= \frac{103^2 \pi}{24000} \text{ (inches}^3\text{)}$$

---

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

$$r = \frac{3}{20} + \frac{h^2}{20}$$

$$r' = \frac{h}{10} \cdot \frac{dh}{dt} = -\frac{1}{5}$$

$$h = 3$$

$$r' = \frac{3}{10} \cdot \frac{dh}{dt} = -\frac{1}{5}$$

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## Question 5

### Overview

In this problem students were presented with a funnel of height 10 inches and circular cross sections. At height $h$ the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $r$ and $h$ are measured in inches. In part (a) students needed to find the average value of the radius of the funnel. This required evaluating $\frac{1}{10}\int_0^{10} \frac{1}{20}\left(3 + h^2\right) dh$ by finding an antiderivative. In part (b) students needed to find the volume of the funnel. By incorporating the fact that the cross sections are circular, the students were expected to set up and evaluate an integral of the form $\pi\int_0^{10} r^2\, dh = \pi\int_0^{10} \left(\left(\frac{1}{20}\right)\left(3 + h^2\right)\right)^2 dh.$ In part (c) students were given that the funnel contains liquid that is draining from the bottom. When the height of the liquid is 3 inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ in/sec. Students were expected to find the rate at which the height is changing at this instant. To solve this related rates problem, students needed to use $r = \frac{1}{20}\left(3 + h^2\right)$ and take the derivative with respect to $t$.

**Sample: 5A**
**Score: 9**

The response earned all 9 points.

**Sample: 5B**
**Score: 6**

The response earned 6 points: 3 points in part (a), no points in part (b), and 3 points in part (c). In part (a) the student's work is correct. In part (b) the student does not present a correct integrand in the integral for volume and did not earn the first point. Without a correct integrand, the student was not eligible for the other points. In part (c) the student's work is correct.

**Sample: 5C**
**Score: 3**

The response earned 3 points: no points in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the student does not present an integral and did not earn the first point. Without presentation of an integral, the student was not eligible for the other points. In part (b) the student presents a correct integrand in the integral for volume and earned the first point. The student does not antidifferentiate correctly and did not earn the second point. The student was not eligible for the third point. In part (c) the student uses the chain rule correctly to find an equation relating $\frac{dh}{dt}$ to $\frac{dr}{dt}$ and earned the first 2 points. The student does not solve for $\frac{dh}{dt}$ and did not earn the third point.

### Question 6

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | $-6$ | 3 | 2 | 8 |
| 2 | 2 | $-2$ | $-3$ | 0 |
| 3 | 8 | 7 | 6 | 2 |
| 6 | 4 | 5 | 3 | $-1$ |

The functions $f$ and $g$ have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of $x$.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of $k$ at $x = 3$.

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

---

(a) $k(3) = f(g(3)) = f(6) = 4$
$k'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot 2 = 5 \cdot 2 = 10$

An equation for the tangent line is $y = 10(x - 3) + 4$.

$3 : \begin{cases} 2 : \text{slope at } x = 3 \\ 1 : \text{equation for tangent line} \end{cases}$

(b)
$$h'(1) = \frac{f(1) \cdot g'(1) - g(1) \cdot f'(1)}{(f(1))^2} = \frac{(-6) \cdot 8 - 2 \cdot 3}{(-6)^2} = \frac{-54}{36} = -\frac{3}{2}$$

$3 : \begin{cases} 2 : \text{expression for } h'(1) \\ 1 : \text{answer} \end{cases}$

(c)
$$\int_1^3 f''(2x)\,dx = \frac{1}{2}\Big[f'(2x)\Big]_1^3 = \frac{1}{2}[f'(6) - f'(2)] = \frac{1}{2}[5 - (-2)] = \frac{7}{2}$$

$3 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

1 of 2

6A

**NO CALCULATOR ALLOWED**

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | −6 | 3 | 2 | 8 |
| 2 | 2 | −2 | −3 | 0 |
| 3 | 8 | 7 | 6 | 2 |
| 6 | 4 | 5 | 3 | −1 |

6. The functions $f$ and $g$ have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of $x$.

   (a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of $k$ at $x = 3$.

$k(3) = f(g(3))$

$= f(6)$

$= 4$

$k'(x) = f'(g(x)) \cdot g'(x)$

$k'(3) = f'(g(3)) \cdot g'(3)$

$= f'(6) \cdot 2$

$= 5 \cdot 2$

$= 10$

$\boxed{y - 4 = 10(x - 3)}$

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Continue problem 6 on page 21.

2 of 2

6A

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

$$h'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{(f(x))^2}$$

$$\boxed{h'(1) = \frac{-3}{2}}$$

$$h'(1) = \frac{f(1)g'(1) - g(1)f'(1)}{(f(1))^2}$$

$$= \frac{(-6)(8) - (2)(3)}{(-6)^2}$$

$$= \frac{-48 - 6}{36}$$

$$= \frac{-54}{36}$$

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

$$\left[\frac{1}{2}f'(2x)\right]_1^3$$

$$\left[\frac{1}{2}f'(6)\right] - \left[\frac{1}{2}f'(2)\right]$$

$$\frac{5}{2} + 1$$

$$\boxed{\frac{7}{2}}$$

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GO ON TO THE NEXT PAGE.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | −6 | 3 | 2 | 8 |
| 2 | 2 | −2 | −3 | 0 |
| 3 | 8 | 7 | 6 | 2 |
| 6 | 4 | 5 | 3 | −1 |

6. The functions $f$ and $g$ have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of $x$.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of $k$ at $x = 3$.

$K'(x) = f'(g(x)) \cdot g'(x)$

$K'(3) = f'(g(3)) \cdot g'(3)$

$K'(3) = f'(6) \cdot 2$

$K'(3) = 5 \cdot 2$

$K'(3) = 10$

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Continue problem 6 on page 21.

(b) Let $h(x) = \frac{g(x)}{f(x)}$. Find $h'(1)$.

$$h'(x) = \frac{f(x)\cdot g'(x) - g(x)\cdot f'(x)}{(f(x))^2}$$

$$\frac{-48-6}{36}$$

$$h'(1) = \frac{(f(1)\cdot g'(1)) - (g(1)\cdot f'(1))}{(f(1))^2}$$

$$\frac{-54}{36}$$

$$\frac{(-6\cdot 8) - (2\cdot 3)}{(-6)^2}$$

---

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

$$\int_1^3 f''(u)\frac{du}{2}$$

$u = 2x$

$du = 2dx$

$\frac{du}{2} = dx$

$$\frac{1}{2}\int_1^3 f''(u)\,du$$

$$\frac{1}{2}\left(f'(3) - f'(1)\right)$$

$$\frac{1}{2}(7-3)$$

$$\frac{4}{2} = 2$$

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1 of 2

6C

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | −6 | 3 | 2 | 8 |
| 2 | 2 | −2 | −3 | 0 |
| 3 | 8 | 7 | 6 | 2 |
| 6 | 4 | 5 | 3 | −1 |

6. The functions $f$ and $g$ have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of $x$.

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of $k$ at $x = 3$.

$K(3) = f(g(3))$

$K(3) = f(6)$

$K(3) = 4$

$\boxed{y - 4 = 4(x-3)}$

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Continue problem 6 on page 21.

2 of 2

6C

(b) Let $h(x) = \frac{g(x)}{f(x)}$. Find $h'(1)$.

$$h'(x) = \frac{g'(x)f(x) - g(x)f'(x)}{[f(x)]^2}$$

$$h'(1) = \frac{g'(1)f(1) - g(1)f'(1)}{\boxed{f(1)}^2}$$

$$= \frac{8 \cdot -6 - 2 \cdot 3}{36} = \frac{-48 - 6}{36} = \frac{54}{36} = \boxed{\frac{8}{6}}$$

(c) Evaluate $\int_1^3 f''(2x)\,dx$.

$$= f'(2x)\Big]_1^3$$

$$= f'(6) - f'(2)$$

$$= 5 - -2 = \boxed{7}$$

$f(x) = -6$

$f(1) = -6 \quad f'(1) = 3$

$f(x) \cdot -6x^3$

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## Question 6

### Overview

In this problem students were given two general functions, $f$ and $g$, that have continuous second derivatives. A table is presented with values of the functions and their derivatives at selected values of $x$. In part (a) students needed to find the equation of the line tangent to the graph of $k$ at $x = 3$, where $k$ is defined by $k(x) = f(g(x))$. This required application of the chain rule and use of values from the table to compute $k'(3) = f'(g(3)) \cdot g'(3)$ and $k(3) = f(g(3))$. In part (b) students were given $h(x) = \dfrac{g(x)}{f(x)}$ and asked to compute $h'(1)$. Students were expected to use the quotient rule and values from the table. Alternately, the product rule and chain rule can be applied to $h(x) = g(x) \cdot (f(x))^{-1}$. In part (c) students needed to evaluate the definite integral $\int_1^3 f''(2x)\,dx$. Using substitution of variables and applying the Fundamental Theorem of Calculus, students were expected to find an antiderivative involving $f'$ and evaluate using values from the table.

**Sample: 6A**
**Score: 9**

The response earned all 9 points.

**Sample: 6B**
**Score: 6**

The response earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In part (a) the student earned the first 2 points. The student does not present an equation for the tangent line. In part (b) the student's work is correct. In part (c) the student has an error with the substitution. The student earned 1 of the first 2 points and was not eligible for the third point.

**Sample: 6C**
**Score: 3**

The response earned 3 points: no points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student does not present a value for $k'(3)$. In part (b) the student earned the first 2 points. The student did not earn the third point because of an error in simplification. In part (c) the student has an error with the constant in the antiderivative. The student earned 1 of the first 2 points and was not eligible for the third point.